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CRITERIA FOR UNSAFE AND SAFE BOUNDARIES OF THE STABILITY DOMAIN FOR EQUATIONS WITH DELAY†

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Using the results of [1–5], criteria are obtained for unsafe and safe segments of the boundaries of the stability domain (BSD) for equilibrium states of first-order equations with delay and systems of second-order equations with delay corresponding to a zero root and a pair of pure imaginary roots. It is shown for an oscillator with delay, unlike one without, that the BSD for its equilibrium state might be unsafe. © 1997 Elsevier Science Ltd. All rights reserved.

Problems involving the determination of unsafe and safe boundaries of the stability domain (BSD) for equilibrium states of systems with delay have been studied in [1–11]. Methods and algorithms for investigating the stability of systems with delay in critical cases, in which they are reduced to truncated systems without delay are given in [6–11]. Formulae have been obtained for a quantity similar to the first Lyapunov quantity for special cases of first-degree equations with delay and the second-degree system considered below [1, 10]. For systems of arbitrary degree with delay, a short form of the criteria for unsafe and safe BSD for the equilibrium states is given in [2, 3]. However, no such simple or convenient criteria for unsafe or safe BSD for the equilibrium states of systems with delay are to be found in [1–11] as for systems without delay in [12].

1. FIRST-DEGREE EQUATIONS WITH DELAY

We will consider how to find the unsafe and safe BSD for the equilibrium state of the equation

$$\dot{x} = ax + \sum_{k=1}^{n} b_k x(t-\tau_k) + F(x, x(t-\tau_1), \ldots, x(t-\tau_n)) \tag{1.1}$$

where x is a scalar, a and b_k are constant coefficients, and $\tau_1 > 0, \tau_2 > 0, \ldots, \tau_n > 0$.

Suppose that the analytic function $F(x_1, x_2, \ldots, x_{n+1})$ is expanded in series in the neighbourhood of the point $x_1 = x_2 = \ldots = x_{n+1}$, starting from terms of no less than second degree in $x_1, x_2, \ldots, x_{n+1}$ of the form

$$F = \sum_{1 \leq i \leq k \leq n+1} a_{ik} x_i x_k + \sum_{1 \leq i \leq k \leq p \leq n+1} a_{ikp} x_i x_k x_p + \ldots$$

where a_{ik}, a_{ikp} are constant coefficients

Suppose that the characteristic equation

$$\Delta(p) = p - a - \sum_{k=1}^{n} b_k e^{-p\tau_k} = 0 \tag{1.2}$$

either has roots $p_{1,2} = \pm i\omega$, or has a root $p_j = 0$ and roots p_i satisfying the condition $\operatorname{Re} p_j < -\sigma < 0$.

We will consider the case where Eq. (1.2) has roots

$$p_{1,2} = \pm i\omega, \quad p_j \text{ with } \operatorname{Re} p_j < -\sigma < 0.$$

We will write Eq. (1.1) in operator form [8]

$$dx_t(\theta)/dt = Lx_t(\theta) + R(x_t(\theta)), \quad x_t(\theta) = x(t+\theta) \tag{1.3}$$

$$Lx_t(\theta) = \begin{cases} \dfrac{dx_t(\theta)}{d\theta}, & -\tau \leq \theta < 0 \\ ax_t(0) + \sum\limits_{k=1}^{n} b_k x_t(-\tau_k), & \theta = 0 \end{cases}$$

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$$R(x_t(\theta)) = \begin{cases} 0, & -\tau \leqslant \theta < 0 \\ F(x_t(0), x_t(-\tau_1)), \dots, x_t(-\tau_n), & \theta = 0 \end{cases}$$

where $\tau = \max(\tau_1, \tau_2, \dots, \tau_n)$, $x(t)$ is a solution of Eq. (1.1) for $t > 0$ with continuously differentiable initial function $x_0(\theta) = \varphi(\theta)$.

We consider the functions

$$b_j(\theta) = \exp\left(\frac{p_j\theta}{\Delta_j}\right), \quad \Delta_j = 1 + \sum_{k=1}^{n} b_k\tau_k \exp(-p_j\tau_k), \quad j = 1,2$$

Using the method described in [8], by means of the change of variables

$$y_j(t) = x_t(0) - \sum_{k=1}^{n} b_k \int_0^{-\tau_k} \exp(-p_j(\tau_k + v))x_t(v)dv$$

$$z_t(\theta) = x_t(\theta) - \sum_{j=1}^{2} b_j(\theta)y_j(t)$$

Eq. (1.3) becomes the "truncated" second-degree system without delay

$$\dot{y}_j = p_j y_j + Q(y_1, y_2), \quad j = 1,2 \tag{1.4}$$

$$Q(y_1,y_2) = F(\psi_1,\psi_2,\dots,\psi_{n+1}) = \sum_{1 \leqslant i \leqslant k \leqslant n+1} a_{ik}\psi_i\psi_k + \sum_{1 \leqslant i \leqslant k \leqslant p \leqslant n+1} a_{ikp}\psi_i\psi_k\psi_p + \dots$$

$$\psi_k = \alpha_{k1}y_1 + \alpha_{k2}y_2 + \sum_{r+q=2} d_{rq}^{(k)} y_1^r y_2^q, \quad k = 1,2,\dots,n+1$$

$$\alpha_{m+1,j} = \exp(-p_j\tau_m)/\Delta_j, \quad j = 1,2; \quad m = 0,1,2,\dots,n; \quad \tau_0 = 0$$

where $d_{rq}^{(k)}$ are constant coefficients defined below.

Multiplying the expressions ψ_i, ψ_k, ψ_p, by one another, we obtain the function $Q(y_1, y_2)$ in the form of a polynomial in (y_1, y_2)

$$Q(y_1,y_2) = \sum_{k \geqslant 2} \sum_{r+q=k} A_{rq} y_1^r y_2^q$$

where

$$A_{20} = \frac{1}{\Delta_1^2} \sum_{1 \leqslant i \leqslant k \leqslant n+1} a_{ik}\sigma_{i1}\sigma_{k1} \tag{1.5}$$

$$A_{11} = \frac{1}{\Delta_1\Delta_2} \sum_{1 \leqslant i \leqslant k \leqslant n+1} a_{ik}(\sigma_{i1}\sigma_{k2} + \sigma_{i2}\sigma_{k1}) \tag{1.6}$$

$$A_{21} = \frac{1}{\Delta_1^2\Delta_2}\left[\sum_{1 \leqslant i \leqslant k \leqslant n+1} a_{ik}(\sigma_{k1}\gamma_{11}^{(i)} + \sigma_{i1}\gamma_{11}^{(k)} + \sigma_{i2}\gamma_{20}^{(k)} + \sigma_{k2}\gamma_{20}^{(i)}) + \right.$$

$$\left. + \sum_{1 \leqslant i \leqslant k \leqslant p \leqslant n+1} a_{ikp}(\sigma_{i1}\sigma_{k1}\sigma_{p2} + \sigma_{i1}\sigma_{k2}\sigma_{p1} + \sigma_{i2}\sigma_{k1}\sigma_{p1})\right]$$

$$\sigma_{m+1,j} = \exp(-p_j\tau_m), \quad j = 1,2, \quad \tau_0 = 0, \quad m = 0,1,2,\dots,n$$

$$\gamma_{rq}^{(1)} = \frac{A_{rq}\Delta_1^r\Delta_2^q}{\Delta(\lambda)}\left(1 - \frac{1}{\Delta_1} - \frac{1}{\Delta_2} + \sum_{k=1}^{n} b_k C_{rq}^{(k)}\right) \tag{1.7}$$

$$\gamma_{rq}^{(k+1)} = \exp(-\lambda\tau_k)\gamma_{rq}^{(1)} + A_{rq}C_{rq}^{(k)}\Delta_1^r\Delta_2^q$$

$$C_{rq}^{(k)} = \sum_{j=1}^{2} \frac{1}{(p_j - \lambda)\Delta_j}(\exp(-p_j\tau_k) - \exp(-\lambda\tau_k))$$

$$\lambda = (r-q)i\omega, \quad r+q = 2, \quad k = 1,2,\dots,n; \quad d_{rq}^{(p)} = \gamma_{rq}^{(p)}/(\Delta_1^r\Delta_2^q)$$

We will not give the form of the remaining coefficients A_{rq} of system (1.4), since they do not appear in the expressions for the quantity similar to the first Lyapunov quantity.

The quantity similar to the first Lyapunov quantity for system (1.4) and, therefore, Eq. (1.1), is found from the formula

$$g = \mathrm{Re}\,A_{21} - \omega^{-1} A_{11}\,\mathrm{Im}(A_{20}) \tag{1.8}$$

In the special case when $n = 1$, where Eq. (1.1) contains only one delay $\tau_1 = \tau$, formulae (1.5) and (1.6) take the simpler form

$$A_{20} = \frac{1}{\Delta_1^2}(a_{11} + a_{12}\delta_1 + a_{22}\delta_1^2) \tag{1.9}$$

$$A_{11} = \frac{1}{\Delta_1\Delta_2}[2(a_{11} + a_{22}) + a_{12}(\delta_1 + \delta_2)] \tag{1.10}$$

$$A_{21} = \frac{1}{\Delta_1^2\Delta_2}[2a_{11}(\gamma_{20}^{(1)} + \gamma_{11}^{(1)}) + 2a_{22}(\delta_2\gamma_{20}^{(2)} + \delta_1\gamma_{11}^{(2)}) +$$

$$+ a_{12}(\gamma_{11}^{(2)} + \gamma_{20}^{(2)} + \delta_1\gamma_{11}^{(1)} + \delta_2\gamma_{20}^{(1)}) + 3(a_{11} + \delta_1 a_{222}) + a_{112}(2\delta_1 + \delta_2) + a_{122}(2 + \delta_2^2)]$$

$$\delta_1 = \exp(-p_1\tau), \quad \delta_2 = \exp(-p_2\tau), \quad \Delta_j = 1 + b_1 e^{-p_j\tau}$$

The quantities $\gamma_{rq}^{(1)}, \gamma_{rq}^{(2)}$ in (1.9), (1.10), are obtained from (1.7) with $n = 1, \tau_1 = \tau, k = 1$.

In the case when $n = 1, \tau_1 = \tau = 1, a = a_{ik} = 0, b_1 = -\pi/2$ in Eq. (1.1), formula (1.8) becomes

$$g = 3a_{111} + a_{122} - \left(\frac{3\pi}{2}a_{222} - \frac{\pi}{2}a_{112}\right)$$

which is the same as the formula obtained for this case in [10].

If $g < 0$, the BSD for Eq. (1.1) is safe; if $g > 0$, it is unsafe [6, 8].

Consider the case when Eq. (1.1) has one zero root. Let

$$a + \sum_{k=1}^{n} b_k = 0, \quad \Delta_0 = \left.\frac{d\Delta(p)}{dp}\right|_{p=0} = 1 + \sum_{k=1}^{n} b_k\tau_k \neq 0$$

Then Eq. (1.2) has the single root $p_1 = 0$. Let the other roots p_j of Eq. (1.2) satisfy the condition $\mathrm{Re}\,p_j < -\sigma < 0$.

Using the method described in [3, 7] in this case, we find quantities similar to the first and second Lyapunov quantities for Eq. (1.1)

$$g_1 = \frac{1}{\Delta_0^2}\sum_{1 \le i \le k \le n+1} a_{ik}, \quad g_2 = \frac{1}{\Delta_0^3}\sum_{1 \le i \le k \le p \le n+1} a_{ikp} \tag{1.11}$$

The form of Eq. (1.1) with $\tau_1 = \tau_2 = \ldots = \tau_n = 0, a + \Sigma_{k=1}^{n} b_k = 0$ implies that the stability of the equilibrium state $x = 0$ depends on the quantity $l_1 = \Sigma_{1 \le i \le k \le n+1} a_{ik}$ if $l_1 \neq 0$, and on $l_2 = \Sigma_{1 \le i \le k \le p \le n+1} a_{ik}$ if $l_1 = 0, l_2 \neq 0$. Assuming that $\Delta_0 \neq 0$, formulae (1.11) imply that if the quantity $l_1 \neq 0$, the boundary Γ: $a + \Sigma_{k=1}^{n} b_k = 0, 1 + \Sigma_{k=1}^{n}\tau_k \neq 0$ of the stability domain for the equilibrium state $x = 0$ of Eq. (1.1) with $\tau_1 = \tau_2 = \ldots = \tau_n = 0$ and any $\tau_1 > 0, \tau_2 > 0, \tau_n > 0$ is unsafe. If $l_1 = 0, l_2 \neq 0$, the boundary Γ with $\tau_1 = \tau_2 = \ldots \tau_n = 0$ and any $\tau_1 > 0, \tau_2 > 0, \ldots, \tau_n > 0$ is of the same kind if $\Delta_0 > 0$, and is different if $\Delta_0 < 0$

In the special case when $n = 1$, where there is only one delay $\tau_1 = \tau$ in Eq. (1.1), the BSD for the equilibrium state $x = 0$ of Eq. (1.1) corresponding to one zero root is the half-segment $a + b_1 = 0, a < 1/\tau$, and the parameter $\Delta_0 = 1 - a\tau > 0$. Thus, if $n = 1$, the boundary $a + b_1 = 0, a < 1/\tau$ is of the same kind if $\tau = 0$ for any $\tau > 0$ in cases where $l_1 \neq 0$ or where $l_1 = 0, l_2 \neq 0$.

Example. We will illustrate how the quantity g is found from formulae (1.7)–(1.10) using the example of Wright's equation [13]

$$\dot{x} = -\alpha x(t-1)(1+x) \tag{1.12}$$

with $\alpha = \alpha_1 = \pi/2$.

Equation (1.12) is a special case of Eq. (1.1), for which $a = 0, b_1 = -\alpha, a_{12} = -\alpha, n = \tau_1 = \tau = 1$.

When $\alpha = \alpha_1$ the characteristic equation

$$\Delta(p) = p + \alpha\exp(-p) = 0$$

has roots $p_{1,2} = \pm i\pi/2$ and p_j with $\mathrm{Re}\,p_j < -\sigma < 0$.

From formulae (1.7), (1.9) and (1.10) we obtain

$$A_{20} = \alpha_1 i / \Delta_1^2, \quad A_{11} = 0$$

$$\gamma_{20}^{(1)} = A_{20}\Delta_1^2(1 - 1/\Delta_1 - 1/\Delta_2 - \alpha_1 C_{20}^{(1)})/\Delta^* \tag{1.13}$$

$$\gamma_{20}^{(2)} = -\gamma_{20}^{(1)} + A_{20}\Delta_1^2 C_{20}^{(1)}, \quad C_{20}^{(1)} = \sum_{j=1}^{2}\frac{1}{(p_j - i\pi)\Delta_j}(\delta_j + 1)$$

$$\Delta^* = -\pi/2 + i\pi, \quad \Delta_1 = 1 + i\pi/2, \quad \Delta_2 = 1 - i\pi/2, \quad \delta_1 = -i, \quad \delta_2 = i$$

$$A_{21} = -\alpha_2(\gamma_{20}^{(2)} + \delta_2\gamma_{20}^{(1)})/(\Delta_1^2\Delta_2) \tag{1.14}$$

From (1.13) and (1.14) with $\alpha = \alpha_1$ using formula (1.18) we obtain the expression

$$g = \mathrm{Re}A_{21} = -(3\pi - 2)/10 < 0$$

which is the same as that obtained in [9, 10].

2. SYSTEMS OF SECOND-DEGREE EQUATIONS WITH DELAY

We will now consider how to determine the unsafe and safe BSD for the equilibrium state of the system

$$\begin{aligned}\dot{x}_l &= a_{l1}x_1 + a_{l2}x_2 + b_{l1}x_1(t-\tau) + b_{l2}x_2(t-\tau) + \\ &+ F_l(x_1, x_2, x_1(t-\tau)), \quad x_2(t-\tau), \quad l = 1, 2, \quad \tau > 0\end{aligned} \tag{2.1}$$

where a_{lk}, b_{lk} $(k = 1, 2)$ are constant coefficients.

Suppose that the analytic functions $F_l(x_1, x_2, x_3, x_4)$ are expanded in series in the neighbourhood of the point $x_1, x_2, x_3, x_4 = 0$ starting with terms of no less than the second degree in x_1, x_2, x_3, x_4, of the following form

$$\begin{aligned}&F_l = F_l^{(2)} + F_l^{(3)} + \ldots, F_l^{(2)} = \sum_{1\leqslant i\leqslant k\leqslant 4} a_{lik}x_i x_k \\ &F_l^{(3)} = \sum_{1\leqslant i\leqslant k\leqslant p\leqslant 4} a_{likp}x_i x_k x_p\end{aligned} \tag{2.2}$$

where a_{lik}, b_{likp} are constant coefficients.

The characteristic equation for the equilibrium state of system (2.1) has the form

$$\Delta(p) = \begin{vmatrix} p - c_{11}(p) & -c_{12}(p) \\ -c_{21}(p) & p - c_{22}(p)\end{vmatrix} = 0 \quad (c_{lk}(p) = a_{lk} + b_{lk}e^{-p\tau}) \tag{2.3}$$

Suppose that Eq. (2.3) has simple roots $p_{1,2} = \pm i\omega$ and roots p_j with $\mathrm{Re}\,p_j < -\sigma < 0$.

Let A and B denote matrices with elements a_{lk} and b_{lk} $(l, k = 1, 2)$ and let $F(x_1, x_2, x_3, x_4)$ denote a vector with components $F_1(x_1, x_2, x_3, x_4)$ and $F_2(x_1, x_2, x_3, x_4)$.

We will write system (2.1) in operator form [8]

$$dx_i(\theta)/dt = Lx_t(\theta) + R(x_t(\theta)), \quad x_t(\theta) = x(t+\theta) \tag{2.4}$$

$$Lx_t(\theta) = \begin{cases} dx_t(\theta)/d\theta, & -r \leqslant \theta < 0 \\ Ax_t(0) + Bx_t(-\tau)), & \theta = 0\end{cases}, \quad R(x_t(\theta)) = \begin{cases} 0, & -\tau \leqslant \theta < 0 \\ F(x_t(0)), \quad x_t(-\tau), & \theta = 0\end{cases}$$

where $x(t)$ is a vector with components $x_1(t), x_2(t)$ which is a solution of system (2.1) with $t > 0$ with a continuously differentiable initial vector function $x_0(\theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$.

Let $\Delta_{21}(p_j)$ be the non-zero cofactors of the element of the second row and first column of the determinants $\Delta(p_j)$ $(j = 1, 2)$.

We will consider vectors $b_j(\theta)$ with components

$$b_j^{(i)}(\theta)=\exp(p_j\theta)\Delta_{2i}(p_j)/\Delta_j,\quad i,j=1,2$$

$$\Delta_j=\Delta_{21}(p_j)d\Delta(p)/dp\Big|_{p=p_j}$$

where $\Delta_{2i}(p_j)$ are the cofactors of the elements of the second row and ith column of the determinants $\Delta(p_j)$.

Using the method described in [8], making the change of variables

$$y_j(t)=\sum_{l=1}^{2}\Delta_{l1}(p_j)\left[x_{lt}(0)+\int_{-\tau}^{0}\exp\left[-p_j(\nu+\tau)\right]\sum_{k=1}^{2}x_{kt}(\nu)b_{lk}d\nu\right]$$

$$z_t(\theta)=x_t(\theta)-\sum_{j=1}^{2}b_j(\theta)y_j(t)$$

where $y_1(t)$ and $y_2(t)$ are scalar variables and $z(t)$ is a vector variable, we transform system (2.4) to the "truncated" second-degree system without delay

$$\dot{y}_j=p_jy_j+Q_j(y_1,y_2),\quad j=1,2 \tag{2.5}$$

$$Q_j(y_1,y_2)=\sum_{l=1}^{2}\Delta_{l1}(p_j)F_l(\psi_1,\psi_2,\psi_3,\psi_4)=$$

$$=\sum_{l=1}\Delta_{l1}(p_j)\left[\sum_{1\leqslant i\leqslant k\leqslant 4}^{2}a_{lik}\psi_i\psi_k+\sum_{1\leqslant i\leqslant p\leqslant 4}a_{likp}\psi_i\psi_k\psi_p+\ldots\right]$$

$$\alpha_{kj}=b_j^{(k)}(0)=\Delta_{2k}(p_j)/\Delta_j(1\leqslant k\leqslant 2)$$

$$\alpha_{kj}=b_j^{(k)}(-\tau)=\exp(-p_j\tau)\alpha_{k-2,j}(3\leqslant k\leqslant 4),\quad j=1,2$$

$$\psi_k=\alpha_{k1}y_1+\alpha_{k2}y_2+\sum_{r+q=2}d_{rq}^{(k)}y_1^ry_2^q,\quad k=1,2,3,4$$

where the variable y_2 is the complex conjugate of y_1 and the constants $d_{rq}^{(k)}$ are defined below.

The functions $Q_j(y_1,y_2)$ can be represented in the form

$$Q_j(y_1,y_2)=\sum_{k=2}^{3}\sum_{r+q=k}A_{rq}^{(j)}y_1^ry_2^q+\ldots,\quad A_{rq}^{(j)}=\sum_{l=1}^{2}\Delta_{l1}(p_j)D_{rq}^{(l)} \tag{2.6}$$

Here

$$D_{11}^{(l)}=2\sum_{i=1}^{4}a_{lii}\quad\alpha_{i1}\alpha_{i2}\quad\sum_{1\leqslant i\leqslant k\leqslant 4}a_{lik}(\alpha_{i1}\alpha_{k2}+\alpha_{i2}\alpha_{k1}) \tag{2.7}$$

$$D_{20}^{(l)}=\sum_{i=1}^{4}a_{lii}\alpha_{i1}^2+\sum_{1\leqslant i<k\leqslant 4}a_{lik}\alpha_{i1}\alpha_{k1},\quad D_{21}^{(l)}=S_1^{(l)}+S_2^{(l)}$$

$$S_1^{(l)}=\sum_{1\leqslant i\leqslant k\leqslant p\leqslant 4}a_{likp}(\alpha_{i1}\alpha_{k1}\alpha_{p2}+\alpha_{i1}\alpha_{k2}\alpha_{p1}+\alpha_{i2}\alpha_{k1}\alpha_{p1})$$

$$S_2^{(l)}=\sum_{1\leqslant i\leqslant k\leqslant 4}a_{lik}(\alpha_{i2}d_{20}^{(k)}+\alpha_{k2}d_{20}^{(i)}+\alpha_{k1}d_{11}^{(i)}+\alpha_{i1}d_{11}^{(k)})$$

The quantities $d_{rq}^{(i)}=d_{rq0}^{(i)}$ $(1\leqslant i\leqslant 2)$, $d_{rq}^{(i)}=d_{rq1}^{(i-2)}$ $(3\leqslant i\leqslant 4)$. The quantities $d_{rq0}^{(l)}$, $d_{rq1}^{(l)}$ are the components of the two-dimensional vectors

$$d_{rq0}=\chi^{-1}(\lambda)\left(D_{rq}-\sum_{j=1}^{2}A_{rq}^{(j)}\alpha_j+BC_{rq}\right),\quad d_{rq1}=e^{-\lambda\tau}d_{rq0}+C_{rq}$$

$$C_{rq}=\sum_{j=1}^{3}A_{rq}^{(j)}\frac{1}{p_j-\lambda}\left(e^{-p_j\tau}-e^{-\lambda\tau}\right)\alpha_j$$

$$\chi(\lambda)=(\lambda E-A-Be^{-\lambda\tau}),\quad \lambda=(r-q)i\omega,\quad r+q=2,\quad q=0,1$$

The vectors D_{rq} and α_j have components $D^{(l)}_{rq}$, $D^{(2)}_{rq}$, and α_{1j}, α_{2j}.

The coefficients $A^{(1)}_{20}, A^{(1)}_{11}, A^{(1)}_{21}$ can be found from (2.6) and (2.7).

The first Lyapunov quantity for system (2.5) can be expressed in terms of the coefficients $A^{(1)}_{20}, A^{(1)}_{11}, A^{(1)}_{21}$ as follows:

$$g = \operatorname{Re} A^{(1)}_{21} - \omega^{-1} A^{(1)}_{11} \operatorname{Im}(A^{(1)}_{20}) \tag{2.8}$$

In the special case where $F_l^{(2)} = 0$ in (2.2), formula (2.8) simplifies to

$$g = \operatorname{Re} A^{(1)}_{21} = \operatorname{Re} \sum_{l-1}^{r} \Delta_{l1}(p_1) S^{(l)}_1$$

For system (2.1) g will be a quantity similar to the first Lyapunov quantity [8].

If $g < 0$, the BSD for system (2.1) is safe; if $g > 0$, it is unsafe.

Note that since system (1.1) is a special case of a system with delay considered in [2], formulae for the coefficients $A^{(j)}_{rq}$ can be obtained from the corresponding formulae in [2].

Note that the quantity g for system (1.1) can also be found from formula (2.8) if the functions $F_l(x_1, x_2, x_3, x_4)$ are only C^r-smooth $r \geqslant 3$.

Example. We will use this criterion to determine the type of BSD for the equilibrium state of an oscillator with delay in the equation

$$\ddot{x}(t) + x(t) = \dot{x}(t-\tau)[\alpha + \beta x(t-\tau) + \gamma x^2(t-\tau)] \tag{2.9}$$

where $\alpha, \beta, \gamma < 0$ are parameters.

This equation is a special case of system (2.1).

The characteristic equation for the equilibrium state $x = 0$ of Eq. (2.9) with $\alpha = 0$ has roots $p_{1,2} = \pm i$.

From formulae (2.6) and (2.7) with $\alpha = 0$, for Eq. (2.9) we find the coefficients

$$A^{(1)}_{20} = \beta\alpha_{31}\alpha_{41}, \quad A^{(1)}_{11} = 0, \quad A^{(1)}_{21} = \beta\left(\alpha_{32}d^{(4)}_{20} + \alpha_{42}d^{(3)}_{20} + \alpha_{31}d^{(4)}_{11} + \alpha_{41}d^{(3)}_{11}\right) +$$

$$+\gamma\left(\alpha^2_{31}\alpha_{42} + 2\alpha_{31}\alpha_{32}\alpha_{42}\right)$$

Here

$$d^{(k)}_{11} = 0, \quad d^{(k)}_{20} = A^{(1)}_{20} \sum_{j=1}^{2} \frac{1}{p_j - 2i}(\exp(-p_j\tau) - \exp(-2i\tau))\alpha_{k-2,j},$$

$$\alpha_{1j} = 1/\Delta_j, \quad \alpha_{2j} = p_j\alpha_{1j}, \quad \alpha_{3j} = \exp(-p_j\tau)/\Delta_j, \quad \alpha_{4j} = p_j\alpha_{3j},$$

$$\Delta_1 = 2i, \quad \Delta_2 = -2i, \quad k = 3,4$$

When $\alpha = 0$ formula (2.8) yields

$$g = \operatorname{Re} A^{(1)}_{21} = \frac{1}{8}\gamma\cos\tau - \frac{1}{24}\beta^2 \sin 3\tau \tag{2.10}$$

Since $\gamma < 0$ by hypothesis, (2.10) shows that the boundary $\alpha = 0$ is safe for $0 \leqslant \tau \leqslant \pi/3$ and becomes unsafe for $\tau = \pi$.

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